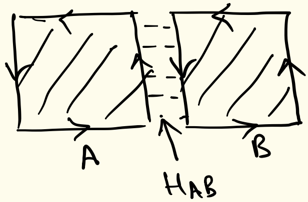
Entanglement Spectrum

Write $\rho_A = e^{-H_A}$, H_A is called entanglement Hamiltonian. For topological phases with chiral edge modes (e.g. quantum Hall systems), there is an interesting relation due to Kitaev-Preskill (2006) and Haldane, Li (2008) that states that physical edge Hamiltonian $= H_A$.

Cut and Glue intuition (Qi, Katsura, Ludwig 2011):



Consider $H_A + H_B + \lambda H_{AB}$ where H_A, H_B denote Hamiltonians of A, B and H_{AB} couples them. At $\lambda = 0$ we have two physical edges (right and left moving). At $\lambda = 1$, one obtains a system without edge modes along the support of H_{AB}. The main point is that any $\lambda > 0$ will gap out the edge modes, therefore $\lambda = 1$ is smoothly connected to infinitesimal λ. Since the bulk is gapped, the problem maps to coupling counter-propagating edge modes of a CFT (Conformal field theory).

An example: Integer QH

$$H := \underbrace{H_L + H_R}_{H_{edge}} + H_{int}$$ } This is a 1+1-d Hamiltonian.

$$H_L = \sum k \, c^+_k c_k \qquad H_R = -\sum k \, d^+_k d_k$$

$$H_{int} = E_g \sum_k [c^+_k d_k + h.c.], \quad E_g = \text{bulk gap.}$$

Since this is a free fermion Hamiltonian, it can be diagonalized exactly. One obtains the following ground state:

$$|G\rangle = e^{-\frac{H_{edge}}{2E_g}} |G_*\rangle$$

where $|G_*\rangle = e^{-\sum_{k>0}[c^+_k d_k + d^+_{-k} c_{-k}]} |G_L\rangle \otimes |G_R\rangle$

where $|G_L\rangle, |G_R\rangle$ are the ground states of H_L, H_R respectively. $|G_*\rangle$ is an equal weight superposition of all quasiparticle excitation states.

$$\boxed{\rho_L = Tr_R |G\rangle\langle G| \simeq e^{-H_L/2E_g}}$$

This discussion generalizes to interacting systems.

One finds,

$$|b\rangle = e^{-\tau_0 H_{edge}} |b_*\rangle$$

where $|b_*\rangle$ is a state that satisfies conformally invariant boundary condition, and has the form:

$$|b_*\rangle = \sum_{n=0}^{\infty} \sum_{j=1}^{d_a(n)} |k(a,n), j; a\rangle_L \otimes |-k(a,n), j, \bar{a}\rangle_R$$

where $k(a,n) = \dfrac{2\pi(h_a+n)}{L}$ denotes the momentum in topological sector 'a' with conformal weight h_a. $\bar{a}$ denotes the conjugate sector with $h_{\bar{a}} = h_a$. $j = 1, 2, \ldots d_a(n)$ labels the elements of orthonormal basis in the subspace of fixed momentum $k(a,n)$.

$$\Rightarrow \rho_L = Tr_R |b\rangle\langle b| \simeq e^{-\tau_0 H_{edge}}.$$

More general discussion: (Peschel, Chung 2011)

Consider $H = H_A + H_B + H_{AB}$

Assuming H_{12} to be the dominant term with unique ground state $|\varphi_0\rangle$, let's calculate the change in the ground state due to H_A, H_B:

$$|\varphi_0'\rangle = |\varphi_0\rangle - \sum_{k \neq 0} |\varphi_k\rangle \frac{\langle \varphi_k | H_A + H_B | \varphi_0 \rangle}{(E_k - E_0)}$$

Let's assume that for states which contribute to this sum, $E_k - E_0 \simeq \Delta$ = indep. of k. Let's also assume that $\langle \varphi_k | H_A | \varphi_0\rangle = \langle \varphi_k | H_B | \varphi_0 \rangle$ (e.g. due to a sym. between A, B.

$$\Rightarrow |\varphi_0'\rangle = |\varphi_0\rangle - \frac{2}{\Delta} \sum_{k \neq 0} |\varphi_k\rangle \langle \varphi_k | H_A | \varphi_0 \rangle$$

$$= |\varphi_0\rangle - \frac{2}{\Delta} H_A' |\varphi_0\rangle$$

where $H_A' = H_A - \langle \varphi_0 | H_A | \varphi_0 \rangle$

$\Rightarrow$ Density matrix for the whole system to $O(1/\Delta)$

$$\rho' = |\varphi_0'\rangle\langle\varphi_0'|$$

$$= |\varphi_0\rangle\langle\varphi_0| - \frac{2}{\Delta}\left[H_A' |\varphi_0\rangle\langle\varphi_0| + |\varphi_0\rangle\langle\varphi_0| H_A' \right]$$

$$= \rho_0 - \frac{2}{\Delta}[H'_A \rho_0 + \rho_0 H'_A]$$ where $\rho_0 = |\psi_0\rangle\langle\psi_0|$

Reduced density matrix for A:

$$\rho'_A = \mathrm{Tr}_B\, \rho$$
$$= \rho^0_A - \frac{2}{\Delta}[H'_A \rho^0_A + \rho^0_A H'_A]$$

where $\rho^0_A = \mathrm{Tr}_B |\psi_0\rangle\langle\psi_0|$.

Finally, let's assume that $\rho^0_A = \mathbb{1}_A$. This is equivalent to saying that the state $|\psi_0\rangle$ is maximally entangled between A and B. This is indeed the case for state $|G_*\rangle$ discussed above in the case of coupled edges.

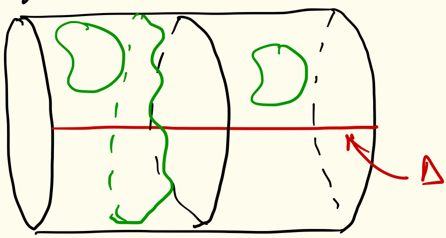
If so, $\rho'_A = \mathbb{1}_A - \frac{4}{\Delta} H'_A$

$$\Rightarrow \boxed{\rho'_A \simeq \frac{e^{-\frac{4}{\Delta} H'_A}}{\mathrm{Tr}\, e^{-4/\Delta\, H'_A}}}$$

Ground state dependence of TEE:

Consider toric code on cylinder:

Two ground states:



$$|\xi_0\rangle = \frac{1}{\sqrt{2N_q}} \sum_{\{q_e\}} |\varphi^A_{\{q_e,0\}}\rangle |\varphi^B_{\{q_e,0\}}\rangle + |\varphi^A_{\{q_e,1\}}\rangle |\varphi^B_{\{q_e,1\}}\rangle$$

Main diff.: correlation between A and B.

$$|\xi_1\rangle = \ldots$$

Consider $|\varphi\rangle = c_0|\xi_0\rangle + c_1|\xi_1\rangle$

$$= \frac{1}{\sqrt{2N_q}} \sum_{\{q_e\}} \left[(c_0 + c_1)|\varphi^A_+\rangle |\varphi^B_+\rangle + (c_0 - c_1)|\varphi^A_-\rangle |\varphi^B_-\rangle \right]$$

$$S_n = \frac{1}{1-n} \log\left[N_\varphi^{1-n} \left\{ \left[\frac{|c_0+c_1|^2}{2}\right]^n \left[\frac{|c_0-c_1|^2}{2}\right]^n \right\}\right]$$

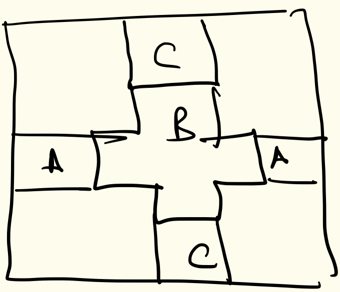
$$\gamma'_n = \log(2) - \frac{1}{1-n} \log\left[p_+^n + p_-^n\right]$$

TEE vanishes for electric field winding eigenstates $|\xi_{0,1}\rangle$ where $c_0 = 0$ or 1

$\Rightarrow$ $p_\pm = \frac{1}{2}$. Max. when $p_+ = 0$ or

$p_- = 0 \Rightarrow$ magnetic eigenstates

Strong subadditivity



$S_{AB} + S_{BC}$

$\geqslant S_B + S_{ABC}$

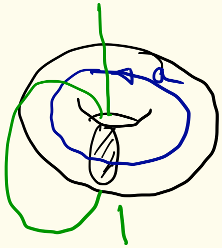
$\gamma_{BC} + \gamma_{AB}$

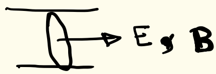
$\leqslant 2\gamma$

Modular S-matrix

$|\psi_a\rangle$ = anyon a along non-contractible path.

$= Z(S^1 \times D^2, a)$





$[E_x, A_y] = 0$

$\langle\psi_a|\psi_b\rangle = Z(S^3$, glued along $S^1 \times S^1)$
contractible direction of one torus is glued to non-contractible cycle of other.

MES :

$$|\Sigma_{1,2}\rangle^x = \frac{1}{\sqrt{2}}\left[|\xi_{00}\rangle \pm |\xi_{01}\rangle\right]$$

$$|\Sigma_{3,4}\rangle^x = \frac{1}{\sqrt{2}}\, |\xi_{10} \pm |\xi_{11}\rangle]$$

$$|\Sigma_1\rangle^y = |\xi_{10}\rangle + |\xi_{01}\rangle$$

$$= \Sigma_3 + \Sigma_4 + |\Sigma_1\rangle + |\Sigma_2\rangle$$

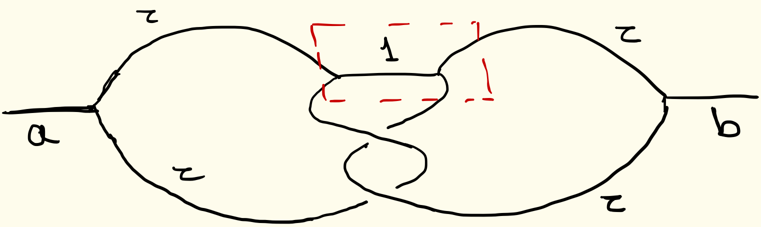
$$|\Sigma_2\rangle^y = \Sigma_1 + \Sigma_2 - \Sigma_3 - \Sigma_4$$
$$= -|\xi_{10}\rangle + |\xi_{00}\rangle$$

$$|\Sigma_3\rangle^y = \Sigma_1 - \Sigma_2 + \Sigma_3 - \Sigma_4$$

$$|\Sigma_4\rangle^y = \Sigma_1 - \Sigma_2 - \Sigma_3 + \Sigma_4$$

2. The diagram is topologically equivalent to

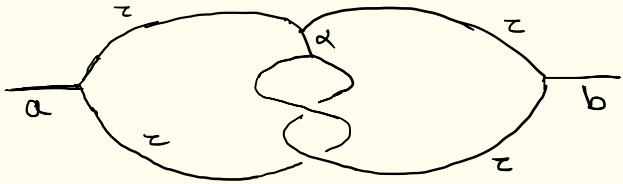
(i)



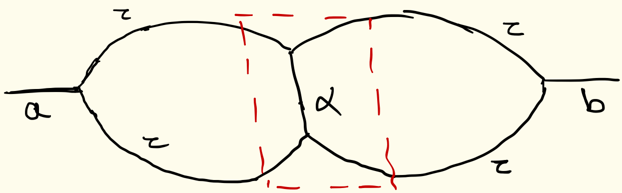
Clearly, if $a \neq b$, then amplitude vanishes. Therefore, the two possibilities are $a = b = 1$, and $a = b = \tau$.

Applying F move to shaded red box above,

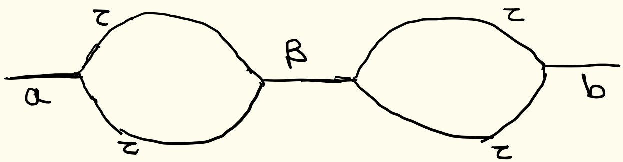
$(i) = F_{1\alpha}$



$= F_{1\alpha} R^2_{\alpha}$



$= F_{1\alpha} R^2_{\alpha} F^{-1}_{\alpha\beta}$



Clearly, $a = \beta = b$ for the diagram to be non-zero.

$\Rightarrow$ prob. for $a = b = \beta = 1$

$$= \left| \left(F \; R^2 \; F^{-1} \right)_{11} \right|^2$$

$$\simeq 0.1459 .$$

Prob. for $a = b = \beta = \tau$

$$= \left| \left(F \; R^2 \; F^{-1} \right)_{1\tau} \right|^2$$

$$\simeq 0.8541 .$$

Recall the braiding matrix derived in lecture 13 was $F R F^{-1}$. Here, we are braiding twice, hence the corresponding unitary is $F R^2 F^{-1}$.